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On the perturbations of a Hamiltonian system

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ABSTRACT

The perturbations of a Hamiltonian system having compounded cycle are studied in this paper. The existence theory and stability theory of singular closed orbits are applied to study the given perturbed systems. By using the small parametric perturbation techniques of differential equations, we study Hopf bifurcation, singular closed orbits bifurcation and give the number and distributions of limit cycles in the above perturbed near Hamiltonian system.

1. Introduction and main results

The 16th of 23 problems posed by D. Hilbert at the Second International Congress of Mathematics, in Paris in 1900 is the problem of the topology of algebraic curves and surfaces and it is still unsolved. In connection with this purely algebraic problem, Hilbert put forward a question, the maximum number and position of Poincarè boundary cycles (limit cycles) for planar polynomial differential equations, which in his eyes, might be attacked by the same method of continuous variation of coefficients. As to the second part of Hilbert's 16th problem, many mathematicians of ordinary differential equations and dynamical systems apply bifurcation methods and qualitative analysis of differential equation to study this problem and get lots of results (see [1,2] for details). In order to obtain limit cycles and their configuration patterns, Li et al. applied the method of detection functions to study the limit cycles in a perturbation of symmetric Hamiltonian systems. 17 limit cycles (resp., 23 limit cycles) and their configuration were found in a quintic Hamiltonian system under the Z_4-equivariant (resp., Z_2-equivariant) perturbation in [3]. In paper [4], 23 limit cycles were found in a Z_3- equivariant quintic planar vector field. Han et al. first used the idea of changing the stability of homoclinic loops to find limit cycles near these loops. Further, this method was developed to investigate the limit cycles bifurcated from singular closed orbits. In [5], the authors studied limit cycles of a quintic planar polynomial vector field and found the above system has 28 limit cycles with two different configurations. A quintic Hamiltonian under 6-order polynomial perturbation is studied in [6] and 35 limit cycles with their distribution are acquired. The study of the limit cycles of Z_q-equivariant quintic planar vector field under $Z_q,\ q = 2, 3, 5, 6$ equivariant perturbations could also be found in [7–9].

In this paper, the following special quintic Hamiltonian

$$\dot{x} = \frac{\partial H}{\partial Y}(x, y),$$
$$\dot{y} = -\frac{\partial H}{\partial x}(x, y), \tag{1}$$

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and its Z_4 invariant quintic polynomial perturbation are considered.

$$\dot{x} = \frac{\partial H}{\partial y}(x, y) + \varepsilon \Phi(x, y, c) \equiv f_1(x, y, c, \varepsilon),$$
$$\dot{y} = -\frac{\partial H}{\partial x}(x, y) + \varepsilon \Psi(x, y, c) \equiv f_2(x, y, c, \varepsilon), \tag{2}$$

where

$$H(x, y) = -8x^2 - \frac{5x^4}{2} - \frac{x^6}{6} - 8y^2 + 15x^2y^2 - \frac{x^4y^2}{2} - \frac{5y^4}{2} - \frac{x^2y^4}{2} - \frac{y^6}{6}, \tag{3}$$

and ε is small positive real number, quintic polynomial functions $\Phi(x, y, c)$, $\Psi(x, y, c)$ are respectively real and imaginary parts of the following complex function

$$F(z, \bar{z}) = c_1 z + c_2 z^2 z + c_3 z^3 \bar{z}^2 + c_4 z\bar{z}^4 + \mathrm{i}c_5 z\bar{z}^4 + c_6 z^5 + \mathrm{i}c_7 z^5, \tag{4}$$

where $z = x + \mathrm{i}y, \bar{z} = x - \mathrm{i}y,\ i^2 = -1,\ x, y \in \mathbb{R}$ and parameter vector $c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7) \in \mathbb{R}^7$.

Let $x = r\cos\theta,\ y = r\sin\theta$, then system (2) is transformed into

$$\begin{cases} \dot{r} = r(c_1 + c_2 r^2 + c_3 r^4)\varepsilon + (c_4 + c_6) r^5 \varepsilon \cos(4\theta) + r^3(10 + c_5 r^2 \varepsilon - c_7 r^2 \varepsilon)\sin(4\theta), \\ \dot{\theta} = 16 + r^4 + r^2(10 + c_5 r^2 \varepsilon + c_7 r^2 \varepsilon)\cos(4\theta) - (c_4 - c_6) r^4 \varepsilon \sin(4\theta). \end{cases} \tag{5}$$

It is easy to check the above system is invariant under coordinate transformation $\theta \rightarrow \theta + \frac{\pi}{2},\ r \rightarrow r$, that is the phase portraits of system (2) is invariant under $\pi/2$ rotation with respect to the origin $O(0, 0)$, and system (2) is called Z_4 equivariant. From [10–12], we know that as $\varepsilon \neq 0$ system (2) is a Hamiltonian if and only if $c_i = 0,\ i = 1, 2, 3$ and $c_4 = -5c_6,\ c_5 = 5c_7$.

System (1) is a Hamiltonian with four saddle points $S_i,\ i = 1, 2, 3, 4$ and five centers $O,\ A_i, i = 1, 2, 3, 4$ and system (1) has the first integral of the form $H(x, y) = h$, where $H(x, y)$ is given in (3). From the fact that system (1) is Z_4 equivariant, we have $H(O) = 0,\ H(A_i) = h_1,\ H(S_i) = h_2,\ i = 1, 2, 3, 4$, where $h_1 = \frac{32}{3},\ h_2 = -\frac{22}{3}$.

The level curves of unperturbed system (1) defined by $H(x, y) = h$ are divided into following categories:

(1) $\Gamma_1^h = \cup_{i=1}^4 \Gamma_{1,i}^h,\ 0 < h \le h_1$, the families of closed orbits $\Gamma_{1,i}^h$ only surrounding the center A_i with anti-clockwise orientation and as h increases to h_1, the closed orbit $\Gamma_{1,i}^h$ approach to singular point $A_i,\ i = 1, 2, 3, 4$.

(2) $\Gamma_2^h = \cup_{i=1}^4 \Gamma_{2,i}^h \cup \Gamma_{2,0}^h,\ h_2 < h \le 0$, the families of closed orbits $\Gamma_{1,i}^h$ only surrounding the center A_i with anti-clockwise orientation and closed orbit $\Gamma_{2,0}^h$ only surrounds the singular point O. As h increases to 0, $\Gamma_{2,0}^h$ approaches to the singular point O.

(3) $\Gamma_3^h = \cup_{i=1}^4 \Gamma_{S_i} \cup \Gamma_{poly},\ h = h_2$, consisting of the four saddle points S_i, four homoclinic loop Γ_{S_i} and one heteroclinic loops $\Gamma_{poly},\ i = 1, 2, 3, 4$. Denote $L_{\widehat{S_i S_j}}$ be saddle connection between saddle point S_i and S_j, then $\Gamma_{poly} = L_{\widehat{S_1 S_2}} \cup L_{\widehat{S_2 S_3}} \cup L_{\widehat{S_3 S_4}} \cup L_{\widehat{S_4 S_1}}$. We call Γ_3^h the compound cycle, and denote it by Γ_{comp}.

(4) $\Gamma_4^h,\ -\infty < h < h_2$, consisting of one family of closed orbit which surrounds all the singular points and as h increases to h_2, Γ_4^h approaches the compound cycle Γ_3^h.

From the above analysis, we plot the phase portraits of system (1) in Fig. 1.

When $0 < \varepsilon \ll 1$, the number of singular points of unperturbed system (1) are well kept. Denoted by $S_i(\varepsilon),\ A_i(\varepsilon)$ the singular points of system (2) near $S_i,\ A_i,\ i = 1, 2, 3, 4$. As the parameter vector satisfies certain conditions, the disturbed system (2) still has four homoclinic loops, four saddle connection loop and compound cycle, respectively denoted by $\Gamma_{S_i(\varepsilon)}$, $\Gamma_{poly}(\varepsilon)$ and $\Gamma_{comp}(\varepsilon)$. By changing the stability of the focus, homoclinic loops and compound cycle of perturbed system (2), we have the following results.

Theorem 1.1. *System (2) has 4 small limit cycles in the neighborhood of $O(0, 0)$, when parameter vector value of system is properly selected.*

Theorem 1.2. *System (2) has 16 small limit cycles, respectively having 4 in the neighborhood of $A_i(\varepsilon),\ i = 1, 2, 3, 4$, when parameter vector value of system is properly selected.*

Theorem 1.3. *System (2) has 4 limit cycles near the inner side of saddle connection loop $\Gamma_{poly}(\varepsilon)$, when parameter vector value of system is properly selected.*

Theorem 1.4. *System (2) has 16 limit cycles, respectively having 4 limit cycles near the inner side of homoclinic loops $\Gamma_{S_i(\varepsilon)}$, $i = 1, 2, 3, 4$, when parameter vector value of system is properly selected.*

Theorem 1.5. *System (2) has 3 large limit cycles circling around all singular points and respectively has 1 limit cycle near the inner side of homoclinic loop $\Gamma_{S_i(\varepsilon)},\ i = 1, 2, 3, 4$, or at least has 3 large limit cycles circling around all singular points and 1 limit cycle near the inner side of saddle connection loop $\Gamma_{poly}(\varepsilon)$, when parameter vector values of system are properly selected.*

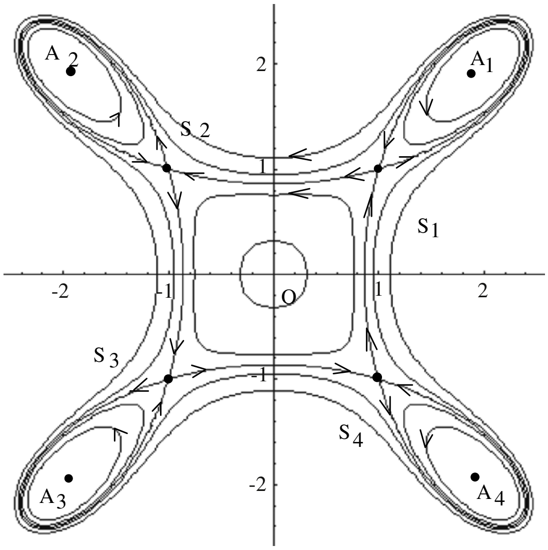

Fig. 1. The phase portraits of unperturbed system (1).

The paper is arranged as follows. In the second part, the Lyapunov constants of focus $O(0, 0)$, $A_i(\varepsilon)$, $i = 1, 2, 3, 4$ of perturbed system (2) are computed. Melnikov functions and quantities which respectively determine the stability of homoclinic loops, four saddle connection loop, and compound cycle are given in the third part. In the last part, the proof of the main results are given.

2. First several Lyapunov constants of system (2)

Consider the following complex form equation

$$\dot{z} = \mathrm{i}z + \sum_{k\geq 2} F_k(z, \bar{z}), \quad z = x + \mathrm{i}y, \ x, y \in \mathbb{R}, \tag{6}$$

where $\bar{z}$ is conjugate of z and $F_k(z, \bar{z})$ are homogeneous polynomials of degree k. We make the change of variables $z = re^{i\theta}$ and transform the above complex equation (6) into

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \mathrm{i}r\,\frac{\dot{z}\bar{z} + z\dot{\bar{z}}}{\dot{z}\bar{z} - z\dot{\bar{z}}} = \frac{\sum\limits_{k\geq 2} r^k \Re(S_k(\theta))}{1 + \sum\limits_{k\geq 1} r^k \Im(S_{k+1}(\theta))}, \tag{7}$$

where $S_k(\theta) = \mathrm{e}^{-\mathrm{i}\theta} F_k(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta})$ and $\Re(\cdot)$, $\Im(\cdot)$ respectively represent real part and imaginary part of complex number. Suppose $r(\theta, r_0)$ is the solution of the above equation satisfying the initial condition $r = r_0$ as $\theta = 0$. Then from analyticity of the above equation, we get $r(\theta, r_0) = r_0 + r_2(\theta)r_0^2 + r_3(\theta)r_0^3 + \cdots$ for r small enough, where $r_k(0) = 0$, $k \geq 2$. It is well known that the first nonzero $r_k(2\pi)$ has an odd subscript $k = 2m + 1$ and we call $V_{2m+1} = r_{2m+1}(2\pi)$ the m-th Lyapunov constant of the origin O.

From [13,14], we know that as $0 < \varepsilon \ll 1$ the stability of singular points O, $A_i(\varepsilon)$ of system (2) are determined by the sign of Lyapunov constants. Let $V_1(P) = (\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y})\,(P)$ be the divergence quantity of the point P and let $V_{2i+1}(P)$, $i = 1, 2, 3, 4$ be the Lyapunov constants of the point P.

The first several Lyapunov constants of the origin O of system (2) are given in the following lemma.

Lemma 2.1. *The divergence quantity and Lyapunov constants of the origin $O(0, 0)$ of system (2) are the followings:*

$$\begin{aligned}
&V_1(O) = 2\varepsilon c_1,\\
&V_3(O) = \frac{1}{8}\pi\varepsilon c_2, \quad \textit{when } V_1(O) = 0,\\
&V_5(O) = \frac{1}{8}\pi\varepsilon c_3, \quad \textit{when } V_1(O) = V_3(O) = 0,\\
&V_7(O) = -\frac{5}{256}\pi\varepsilon(c_4 + 5c_6), \quad \textit{when } V_i(O) = 0,\ i = 1, 3, 5,\\
&V_9(O) = -\frac{1}{128}\pi c_6\varepsilon^2(c_5 - 5c_7), \quad \textit{when } V_i(O) = 0,\ i = 1, 3, 5, 7.
\end{aligned} \tag{8}$$

Proof. First we make time scale $\tau = \frac{1}{16}t$ transform system (2) to norm form in the neighborhood of the origin $O(0,0)$. Then write the transformed system into the following complex form by the coordinate change $x = (z+\bar{z})/2,\ y = -i/2(z-\bar{z})$:

$$\dot{z} = \mathrm{i}z + \frac{1}{16}\mathrm{i}z^3\bar{z}^2 + \frac{5}{8}\mathrm{i}\bar{z}^3 + \varepsilon\left[\frac{a_3}{16}z + \frac{b_6}{16}z^5 + \frac{a_4}{16}z^2\bar{z} + \frac{a_5}{16}z^3\bar{z}^2 + \frac{a_2}{16}z\bar{z}^4 + \frac{b_2}{16}z\bar{z}^4\right]. \tag{9}$$

By applying the formula and method of deducing the first several Lyapunov constants given in [14], we get the conclusion of the lemma.

To determine the stability of singular points $A_i(\varepsilon)$ and stability of compound cycle, we need the first several Lyapunov constants of $A_i(\varepsilon)$ and the saddle quantity of $S_i(\varepsilon)$. Noting the fact that system (2) is Z_4 equivariant, we only need to compute Lyapunov constants of $A_1(\varepsilon)$ and saddle quantity of $S_1(\varepsilon)$. First, we get the asymptotic expressions of $A_1(\varepsilon),\ S_1(\varepsilon)$ of perturbed system (2) as $\varepsilon \neq 0$ and small in the following lemma. □

Lemma 2.2. *When $0 < \varepsilon \ll 1$, the asymptotic expressions of $A_1(\varepsilon),\ S_1(\varepsilon)$ of the perturbed system (2) are the followings:*

$$\begin{aligned} &A_1(\varepsilon) = A_1(0) + \varepsilon A_1'(0) + O(\varepsilon^2),\\ &S_1(\varepsilon) = S_1(0) + \varepsilon S_1'(0) + O(\varepsilon^2), \end{aligned} \tag{10}$$

where

$$\begin{aligned} &A_1(0) = (2,2), \qquad S_1(0) = (1,1), \qquad A_1'(0) = \frac{1}{480}(a_1, a_2), \qquad S_1'(0) = \frac{1}{240}(b_1,\ b_2),\\ &a_1 = -3c_1 - 24c_2 - 192c_3 + 192c_4 + 640c_5 + 192c_6 + 640c_7,\\ &a_2 = 3c_1 + 24c_2 + 192c_3 - 192c_4 + 640c_5 - 192c_6 + 640c_7,\\ &b_1 = -3c_1 - 6c_2 - 12c_3 + 12c_4 - 40c_5 + 12c_6 - 40c_7,\\ &b_2 = 3c_1 + 6c_2 + 12c_3 - 12c_4 - 40c_5 - 12c_6 - 40c_7. \end{aligned}$$

Proof. Substitute the expression of (10) into the following equations:

$$H_y(x,y) + \varepsilon\Phi(x,y,c) = 0, \tag{11}$$

$$-H_x(x,y) + \varepsilon\Psi(x,y,\varepsilon) = 0. \tag{12}$$

By combining the like terms as to the variable ε, balancing the coefficients before $\varepsilon^k,\ k = 0, 1, 2$, and solving the acquire equations, we get (11).

From the above Lemma 2.2, we first move $A_1(\varepsilon)$ to the origin by a coordinate change, then applying the same process of Lemma 2.1, we get the following first Lyapunov constants of $A_1(\varepsilon)$ in the following lemma. □

Lemma 2.3. *The divergence quantity and Lyapunov constants of system (2) at $A_1(\varepsilon)$ are the followings:*

$$\begin{aligned} &V_1(A_1(\varepsilon)) = (2c_1 + 32c_2 + 384c_3 - 128c_4 - 640c_6)\varepsilon + O(\varepsilon^2),\\ &V_3(A_1(\varepsilon)) = \frac{\pi}{78\sqrt{30}}(-35c_2 - 660c_3 + 256c_4 + 1280c_6)\varepsilon + O(\varepsilon^2), \quad \text{when } V_1(A_1(\varepsilon)) = 0,\\ &V_5(A_1(\varepsilon)) = \frac{\pi}{3459\sqrt{30}}(5665c_3 - 4099c_4 - 20495c_6)\varepsilon + O(\varepsilon^2), \quad \text{when } V_i(A_1(\varepsilon)) = 0,\ i = 1,3,\\ &V_7(A_1(\varepsilon)) = \frac{60053455}{972594324}\sqrt{\frac{5}{6}}\pi\, c_3\,\varepsilon + O(\varepsilon^2), \quad \text{when } V_i(A_1(\varepsilon)) = 0,\ i = 1,3,5,\\ &V_9(A_1(\varepsilon)) = O(\varepsilon^2), \quad \text{when } V_i(A_1(\varepsilon)) = 0,\ i = 1,3,5,7. \end{aligned}$$

3. Melnikov functions and stability quantities

Suppose $L_{\widehat{S_iS_j}}$ is a saddle connection which connects the saddle points S_i with S_j of system (1), $i, j = 1, 2, 3, 4$. Generally speaking, when $\varepsilon \neq 0$, saddle connections $L_{\widehat{S_iS_j}}$ and homoclinic loop Γ_{S_i} of system (2) will be broken and the curve connecting the saddle points will be altered. To study the existence of saddle connection between two given saddle points $S_i(\varepsilon),\ S_j(\varepsilon)$ of perturbed system (2), we choose $M_1 \in L_{\widehat{S_iS_j}}$ and let l_1 be a segment normal to $L_{\widehat{S_iS_j}}$ at point M_1. For $|\varepsilon| \ll 1$ the line l_1 transvertically intersects with $L^s_{S_iS_j}(\varepsilon),\ L^u_{S_i,S_j}(\varepsilon)$ at points $M_1^s(\varepsilon),\ M_1^u(\varepsilon)$ respectively, where $L^s_{S_i,S_j}(\varepsilon)$ and $L^u_{S_i,S_j}(\varepsilon)$ are saddle separatrix near L_{S_i,S_j} satisfying $\omega(L^s_{S_i,S_j}(\varepsilon)) = S_j(\varepsilon), \alpha(L^u_{S_i,S_j}(\varepsilon)) = S_i(\varepsilon)$.

Let

$$d(\varepsilon, L_{\widehat{S_iS_j}}) = -\vec{n}_1 \cdot \overrightarrow{M_1^u(\varepsilon)M_1^s(\varepsilon)}, \tag{13}$$

where $\vec{n}_1 = (H_x(M_1), H_y(M_1))/\|(H_y(M_1), -H_x(M_1))\|$. Therefore, the directed distance between $L^s_{S_i,S_j}(\varepsilon)$ and $L^u_{S_i,S_j}(\varepsilon)$ can be measured by $d(\varepsilon, L_{\widehat{S_i S_j}})$. Similarly, the distance between the stable manifold and unstable manifold of saddle point $S_i(\varepsilon)$ of system (2) can be measured by $d(\varepsilon, \Gamma_{S_i})$.

From [15], we know that

$$d(\varepsilon, L_{\widehat{S_i S_j}}) = \varepsilon M(L_{\widehat{S_i S_j}}) + O(\varepsilon^2), \qquad d(\varepsilon, \Gamma_{S_i}) = \varepsilon M(\Gamma_{S_i}) + O(\varepsilon^2), \tag{14}$$

where $M(L_{\widehat{S_i S_j}})$ (*res* $M(\Gamma_{S_i})$) is called Melnikov function of saddle connection $L_{\widehat{S_i S_j}}$ (res Γ_{S_i}).

Noting that system (2) is Z_4-equivariant, that is, the phase portraits of system (2) are invariant under the rotation of $\pi/2$ around the origin $O(0, 0)$, then we have the following remark.

Remark 1. The Melnikov functions of saddle connections of system (2) satisfy the following equations:

$$\begin{aligned} &M(\Gamma_{S_1}) = M(\Gamma_{S_2}) = M(\Gamma_{S_3}) = M(\Gamma_{S_4}), \\ &M(L_{\widehat{S_1 S_2}}) = M(L_{\widehat{S_2 S_3}}) = M(L_{\widehat{S_3 S_4}}) = M(L_{\widehat{S_4 S_1}}). \end{aligned} \tag{15}$$

From the above remark, for simplicity we only discuss the case of the saddle connection $L_{\widehat{S_1 S_2}}$ and Γ_{S_1} of system (1) under perturbations. As to the expression of $M(L_{\widehat{S_1 S_2}})$ and $M(\Gamma_{S_1})$, we have the following lemma.

Lemma 3.1. *For $\varepsilon \neq 0$ small, the Melnikov functions $M(L_{\widehat{S_1 S_2}})$ and $M(\Gamma_{S_1})$ respectively has the following form:*

$$\begin{aligned} &M(L_{\widehat{S_1 S_2}}) \approx -1.62165c_1 - 1.83948c_2 - 2.3055c_3 + 0.446973c_4 + 2.23487c_6, \\ &M(\Gamma_{S_1}) \approx -1.86957c_1 - 23.8758c_2 - 255.665c_3 + 82.3837c_4 + 411.918c_6, \end{aligned} \tag{16}$$

and $N_i > 0,\ i = 1, 2$ are constants.

Proof. From [15] and noticing that unperturbed system (1) is a Hamiltonian system, we have

$$\begin{aligned} &M(\Gamma_{S_1}) = \int_{\Gamma_{S_1}} \Psi(x, y, c)\mathrm{d}x - \int_{\Gamma_{S_1}} \Phi(x, y, c)\mathrm{d}y, \\ &M(L_{\widehat{S_1 S_2}}) = \int_{L_{\widehat{S_1 S_2}}} \Psi(x, y, c)\mathrm{d}x - \int_{L_{\widehat{S_1 S_2}}} \Phi(x, y, c)\mathrm{d}y. \end{aligned} \tag{17}$$

By using Mathematics 4.0, we obtain the following equations of saddle connections $L_{\widehat{S_1 S_2}}:\ y = y_{1,2}(x),\ -1 \le x \le 1$; $x = x_{1,2}(y) \ge 0,\ y_0 \le y \le 1,\ x = -x_{1,2}(y),\ y_0 \le y \le 1$; $\Gamma_{S_1}:\ y = \hat{y}_{S_1}(x),\ 1 \le x \le y_1,\ y = \tilde{y}_{S_1}(x) \le \hat{y}_{S_1}(x),\ 1 \le x \le y_1$; $x = \hat{x}_{S_1}(y),\ 1 \le y \le y_1,\ x = \tilde{x}_{S_1}(y) \le \hat{x}_{S_1}(y),\ 1 \le y \le y_1$, all of which are implicitly determined by the equation $H(x, y) = -\frac{22}{3}$ and $y_0 \approx 0.859105,\ y_1 \approx 2.40808$.

$$\begin{aligned} &\int_{L_{\widehat{S_1 S_2}}} \Psi(x, y, c)\mathrm{d}x = \int_1^{-1} \Psi(x, y_{1,2}(x), c)\mathrm{d}x = \sum_{i=1}^{7} c_i k_{i,1}, \\ &\int_{L_{\widehat{S_1 S_2}}} \Phi(x, y, c)\mathrm{d}y = \int_1^{y_0} \Phi(x_{1,2}(y), y, c)\mathrm{d}y + \int_{y_0}^{1} \Phi(-x_{1,2}(y), y, c)\mathrm{d}y = \sum_{i=1}^{7} c_i k_{i,2}, \\ &\int_{\Gamma_{S_1}} \Psi(x, y, c)\mathrm{d}x = \int_1^{y_1} \Psi(x, \tilde{y}_{S_1}(x), c)\mathrm{d}x + \int_{y_1}^{1} \Psi(x, \hat{y}_{S_1}(x), c)\mathrm{d}x = \sum_{i=1}^{7} c_i k_{i,3}, \\ &\int_{\Gamma_{S_1}} \Phi(x, y, c)\mathrm{d}y = \int_1^{y_1} \Phi(\tilde{x}_{S_1}(y), y, c)\mathrm{d}y + \int_{y_1}^{1} \Phi(\hat{x}_{S_1}(y), y, c)\mathrm{d}y = \sum_{i=1}^{7} c_i k_{i,4}. \end{aligned}$$

By using numeric integral computation, we get the following numeric results:

$$\begin{aligned} &k_{1,1} \approx -1.81082, \quad k_{2,1} \approx -2.12257, \quad k_{3,1} \approx -2.74974, \quad k_{4,1} \approx 1.02951, \\ &k_{5,1} = 0, \quad k_{6,1} \approx 2.45077, \quad k_{7,1} = 0, \\ &k_{1,2} \approx -0.189176, \quad k_{2,2} \approx -0.283081, \ k_{3,2} \approx -0.444235, \quad k_{4,2} \approx 0.58254, \\ &k_{5,2} = 0, \quad k_{6,2} \approx 0.215901, \quad k_{7,2} = 0, \\ &k_{1,3} \approx -0.934783, \quad k_{2,3} \approx -11.9379, \quad k_{3,3} \approx -127.832, \quad k_{4,3} \approx 41.1918, \\ &k_{5,3} \approx 169.012, \quad k_{6,3} \approx 205.959, \quad k_{7,3} = 0, \\ &k_{1,4} \approx 0.934783, \quad k_{2,4} \approx 11.9379, \quad k_{3,4} \approx 127.832, \quad k_{4,4} \approx -41.1918, \\ &k_{5,4} \approx 169.012, \quad k_{6,4} \approx -205.959, \quad k_{7,4} = 0. \end{aligned}$$

From the above numeric results and (17), we get the expressions of Melnikov functions $M(L_{\widehat{S_1S_2}})$ and $M(\Gamma_{S_1})$. The proof is completed. □

Obviously, a saddle connection of system (2) appears near $L_{\widehat{S_iS_j}}$ for $\varepsilon \neq 0$ small if and only if $d(\varepsilon, L_{\widehat{S_iS_j}}) = 0$. Therefore, we have the following lemma.

Lemma 3.2. *There exists a function*

$$\varphi_1(c_2, c_3, c_4, c_6, \varepsilon) \approx -1.13433c_2 - 1.4217c_3 + 0.275629c_4 + 1.37814c_6 + O(\varepsilon) \tag{18}$$

such that system (2) has a saddle connection loop $\Gamma_{poly}(\varepsilon) = L_{\widehat{S_1(\varepsilon)S_2(\varepsilon)}} \cup L_{\widehat{S_2(\varepsilon)S_3(\varepsilon)}} \cup L_{\widehat{S_3(\varepsilon)S_4(\varepsilon)}} \cup L_{\widehat{S_4(\varepsilon)S_1(\varepsilon)}}$ near $L_{poly} = L_{\widehat{S_1S_2}} \cup L_{\widehat{S_2S_3}} \cup L_{\widehat{S_3S_4}} \cup L_{\widehat{S_4S_1}}$ if and only if $c_1 = \varphi_1(c_2, c_3, c_4, c_6, \varepsilon)$.

Suppose A is a point in the inner side of $\Gamma_{poly}(\varepsilon)$ of system (2). If ω-set of A is $\Gamma_{poly}(\varepsilon)$, then we call $\Gamma_{poly}(\varepsilon)$ is isolated and stable; If α-set of A is $\Gamma_{poly}(\varepsilon)$, then we call $\Gamma_{poly}(\varepsilon)$ is unstable.

As to the stability of $\Gamma_{poly}(\varepsilon)$ of system (2), we have the following lemma.

Lemma 3.3. *Let $c_1 = \varphi_1(c_2, c_3, c_4, c_6, \varepsilon)$.*

(1) *There exists a function $\psi_1(c_3, c_4, c_6, \varepsilon) \approx -3.69139c_3 + 1.29965c_4 + 6.49826c_6 + O(\varepsilon)$ such that when $c_2 - \psi_1(c_3, c_4, c_6, \varepsilon) < 0 (> 0)$, then the saddle connection loop $\Gamma_{poly}(\varepsilon)$ of system (1) is stable (unstable).*
(2) *If $c_2 - \psi_1(c_3, c_4, c_6, \varepsilon) = 0$, the there exists a function $\psi_2(c_4, c_6, \varepsilon) \approx 0.540826c_4 + 2.70413c_6 + O(\varepsilon)$ such that when $c_3 - \psi_2(c_4, c_6, \varepsilon) > 0 (< 0)$, then $\Gamma_{poly}(\varepsilon)$ is stable (unstable).*
(3) *If $c_2 = \psi_1(c_3, c_4, c_6, \varepsilon)$ and $c_3 = \psi_2(c_4, c_6, \varepsilon)$, then there exists a function $\psi_3(c_6, \varepsilon) = -5c_6 + O(\varepsilon)$, and if $c_4 - \psi_3(c_6, \varepsilon) > 0 (< 0)$, then $\Gamma_{poly}(\varepsilon)$ is stable (unstable).*

Proof. The condition of lemma and the Z_4-equivariance of system (2) assure that system (2) has a saddle connection loop $\Gamma_{poly}(\varepsilon)$. Denote $div(S_1(\varepsilon)) = [\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}]_{(x,y)=S_1(\varepsilon)}$ the divergence quantity of $S_1(\varepsilon)$ and by direct computing, we get

$$div(S_1(\varepsilon)) \approx (5.73134c_2 + 21.1566c_3 - 7.44874c_4 - 37.2437c_6)\varepsilon + O(\varepsilon^2).$$

Let $div(S_1(\varepsilon)) = 0$ and apply the implicit theorem, we have $c_2 = \psi_1(c_3, c_4, c_6, \varepsilon)$. From the results of [16–18], we know that the stability of $\Gamma_{poly}(\varepsilon)$ is determined by the sign of $div(S_1(\varepsilon))$. Then first part of the lemma follows.

When $c_2 = \psi_1(c_3, c_4, c_6, \varepsilon)$, that means $\Gamma_{poly}(\varepsilon)$ is degenerated. Denote $L_{\widehat{S_1(\varepsilon)S_2(\varepsilon)}}$ the saddle connection between $S_1(\varepsilon)$ and $S_2(\varepsilon)$ of system (1). Then the determination of stability of $\Gamma_{poly}(\varepsilon)$ must resolve to sign of the following quantity

$$\begin{aligned}\sigma(L_{\widehat{S_1(\varepsilon)S_2(\varepsilon)}}) &= \oint_{L_{\widehat{S_1(\varepsilon)S_2(\varepsilon)}}} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) dt \\ &= \varepsilon\left[\oint_{L_{\widehat{S_1S_2}}} \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}\right)\frac{dx}{H_y} + O(\varepsilon)\right] \approx (-0.499966c_4 + 0.270395c_4 + 1.35197c_6)\varepsilon + O(\varepsilon^2). \end{aligned} \tag{19}$$

Let $\sigma(L_{\widehat{S_1(\varepsilon)S_2(\varepsilon)}}) = 0$, then we get $c_3 = \psi_2(c_4, c_6, \varepsilon)$. From relationship between the sign of $\sigma(L_{\widehat{S_1(\varepsilon)S_2(\varepsilon)}})$ and the inner stability of saddle connection loop $\Gamma_{poly}(\varepsilon)$, we know the second part of the lemma holds.

When $\Gamma_{poly}(\varepsilon)$ of system (1) is more degenerated, that is $div(S_1(\varepsilon)) = 0$ and $\sigma(L_{\widehat{S_1(\varepsilon),S_2(\varepsilon)}}) = 0$, from [18], we know that the inner stability of $\Gamma_{poly}(\varepsilon)$ is determined by the saddle quantity of $S_1(\varepsilon)$. Denote $R_1(S_1(\varepsilon))$ the first saddle quantity of $S_1(\varepsilon)$ of system (2). To derive the formulae of $R_1(S_1(\varepsilon))$, first from Lemma 2.2 we move the saddle point $S_1(\varepsilon)$ to the origin $O(0, 0)$. Then make a time scale and a coordinate change of the form

$$x = a_3y_1 + b_3y_2, \quad y = y_1 + y_2, \qquad t = a_4\tau, \tag{20}$$

to transform the system to the Jordan form, where $a_3 = \frac{1}{7}(13 + 2\sqrt{30}) + \frac{1}{196}[2(13 + 2\sqrt{30})(2c_1 + 7c_2 + 20c_3 + 2c_5 - 40c_6 + 10c_7) + 7(16c_5 + \sqrt{\frac{2}{15}}(59c_5 + 35c_7))]\varepsilon + O(\varepsilon^2)$,

$b_3 = \frac{1}{7}(13-2\sqrt{30}) + \frac{1}{196}[-2(-13+2\sqrt{30})(2c_1+7c_2+20c_3+2c_5-40c_6+10c_7)+7(16c_5-\sqrt{\frac{2}{15}}(59c_5+35c_7))]\varepsilon + O(\varepsilon^2)$,

$a_4 = 8\sqrt{30} + (59\sqrt{\frac{2}{15}}c_5 + 7\sqrt{\frac{10}{3}}c_7)\varepsilon + O(\varepsilon^2)$.

Finally by applying the formula of first order saddle quantity given in [19,15], we get $R_1(S_1(\varepsilon)) = -(1.12419c_4 + 5.62093c_6)\varepsilon + O(\varepsilon^2)$. Let $R_1(S_1(\varepsilon)) = 0$, we get the expression of $\psi_3(c_6, \varepsilon) = -5c_6 + O(\varepsilon)$. Noticing the closed saddle connection loop $\Gamma_{poly}(\varepsilon)$ is oriented counter-clockwise and using the result in [18], the lemma is proved.

Similarly, as $0 < \varepsilon \ll 1$, we have the following lemma. □

Lemma 3.4. *There exists a function* $\varphi_2(c_2, c_3, c_4, c_6, \varepsilon) \approx -12.7708c_2 - 136.751c_3 + 44.0657c_4 + 220.328c_6 + O(\varepsilon)$ *such that system* (2) *has homoclinic loop* $\Gamma_{S_1(\varepsilon)}$ *near* Γ_{S_1} *if and only if* $c_1 = \varphi_2(c_2, c_3, c_4, c_6, \varepsilon)$. *Further suppose the homoclinic loop* $\Gamma_{S_1(\varepsilon)}$ *of system* (2) *appears.*

(1) *There exists a function* $\psi_4(c_3, c_4, c_6, \varepsilon) \approx -14.2235c_3 + 4.56809c_4 + 22.8404c_6 + O(\varepsilon)$ *such that homoclinic loop* $\Gamma_{S_1(\varepsilon)}$ *is stable (unstable) as* $c_2 - \psi_4(c_3, c_4, c_6, \varepsilon) > 0(< 0)$.
(2) *If* $c_2 - \psi_4(c_3, c_4, c_6, \varepsilon) = 0$, *then there exists a function* $\psi_5(c_4, c_6, \varepsilon) \approx 0.49496c_4 + 2.4748c_6 + O(\varepsilon)$ *such that homoclinic loop* $\Gamma_{S_1(\varepsilon)}$ *is stable (unstable) as* $c_3 - \psi_5(\varepsilon) > 0(< 0)$.
(3) *If* $c_2 - \psi_4(c_3, c_4, c_6, \varepsilon) = 0,\ c_3 = \psi_5(c_4, c_6, \varepsilon)$, *then there exists a function* $\psi_6(c_6, \varepsilon) = -5c_6 + O(\varepsilon)$ *such that homoclinic loop* $\Gamma_{S_1(\varepsilon)}$ *is stable (unstable) as* $c_4 - \psi_6(\varepsilon) < 0(> 0)$.

Proof. From Lemma 3.1, we know the expression of $M(\Gamma_{S_1})$ and let $d(\varepsilon, \Gamma_{S_1}) = 0$, then we get $c_1 = \varphi_2(c_2, c_3, c_4, c_6, \varepsilon)$. From the definition of $d(\varepsilon, \Gamma_{S_1})$, we know that the first part of the lemma is true.

Furthermore, from $div(S_1(\varepsilon)) = 0$, we get $c_2 = \psi_4$.

If $c_2 = \psi_4$, from [17,18,20], we know that the inner stability of homoclinic loop $\Gamma_{S_1(\varepsilon)}$ is determined by the sign of the integral of the following divergence quantity:

$$\begin{aligned}\sigma(\Gamma_{S_1(\varepsilon)}) &= \oint_{\Gamma_{S_1(\varepsilon)}} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) \mathrm{d}t \\ &= \varepsilon\left[\oint_{\Gamma_{S_1(\varepsilon)}} \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}\right)\frac{\mathrm{d}x}{H_y} + O(\varepsilon)\right] \\ &\approx (-1.27734c_3 + 0.632233c_4 + 3.16117c_6)\varepsilon + O(\varepsilon^2). \end{aligned} \tag{21}$$

Let $\sigma(\Gamma_{S_1(\varepsilon)}) = 0$, we get $c_3 = \psi_5$. By using a similar analysis to Lemma 3.2, we know that the remainder of the lemma is also true.

As we know, unperturbed system (1) is a Hamiltonian system and has a compound cycle Γ_{comp}. It is easy to know that a compound cycle of system (2) appears if and only if $d(\varepsilon, \Gamma_{S_1(\varepsilon)}) = d(\varepsilon, L_{\widehat{S_1 S_2}}) = 0$. We denote the compound cycle of system (2) by $\Gamma_{comp}(\varepsilon)$, that is $\Gamma_{comp}(\varepsilon) = \cup_{i=1}^{4} \Gamma_{S_i(\varepsilon)} \cup \Gamma_{poly}(\varepsilon)$. Suppose a point B is in the outer side of $\Gamma_{comp}(\varepsilon)$, if $\omega(B) = \Gamma_{comp}(\varepsilon)$, then we call the compound cycle $\Gamma_{comp}(\varepsilon)$ outer stable; if $\alpha(B) = \Gamma_{comp}(\varepsilon)$, then we call $\Gamma_{comp}(\varepsilon)$ outer unstable.

As to the outer stability of the compound cycle $\Gamma_{comp}(\varepsilon)$ of system (2), we have the following lemma. □

Lemma 3.5. *There exist functions* $\varphi_3(c_3, c_4, c_6, \varepsilon) \approx 11.7703c_3 - 3.99306c_4 - 19.9635c_6 + O(\varepsilon)$ *and* $\varphi_4(c_3, c_4, c_6, \varepsilon) \approx -11.6298c_3 + 3.76318c_4 + 18.8159c_6 + O(\varepsilon)$ *such that system* (2) *has compound cycle* $\Gamma_{comp}(\varepsilon)$ *if and only if* $c_1 = \varphi_3(c_3, c_4, c_6, \varepsilon),\ c_2 = \varphi_4(c_3, c_4, c_6, \varepsilon)$. *Furthermore suppose compound cycle* $\Gamma_{comp}(\varepsilon)$ *of system* (2) *appears.*

(1) *There exists a function* $\psi_7(c_4, c_6, \varepsilon) \approx 0.310331c_4 + 1.55166c_6 + O(\varepsilon)$ *such that* $\Gamma_{comp}(\varepsilon)$ *is outer stable(unstable) as* $c_3 - \psi_7(c_4, c_6, \varepsilon) > 0(< 0)$.
(2) *If* $c_3 = \psi_7(c_4, c_6, \varepsilon)$, *then there exists a function* $\psi_8(c_6, \varepsilon) = -5c_6 + O(\varepsilon)$ *such that* $\Gamma_{comp}(\varepsilon)$ *is outer stable(unstable) as* $c_4 - \psi_8(c_6, \varepsilon) > 0(< 0)$.

Proof. Let $d(\varepsilon, \Gamma_{S_1(\varepsilon)}) = d(\varepsilon, L_{\widehat{S_1 S_2}}) = 0$, then from Implicit Function Theorem we get $c_1 = \varphi_3(c_3, c_4, c_6, \varepsilon),\ c_2 = \varphi_4(c_3, c_4, c_6, \varepsilon)$. Furthermore, from $div(S_1(\varepsilon)) = 0$, we get $c_3 = \psi_7(c_4, c_6, \varepsilon)$.

If $c_3 = \psi_7(c_4, c_6, \varepsilon)$, then from $\sigma(\Gamma_{comp}(\varepsilon)) = \sigma(\Gamma_{S_1(\varepsilon)}) + \sigma(L_{\widehat{S_1(\varepsilon) S_2(\varepsilon)}}) = 0$, we get $c_4 = \psi_8(c_6, \varepsilon)$.

From [18] or the proof of the main result in [20], we know the conclusions of lemma hold. □

4. Proof of main results

In the following, we always assume that parameters $c_5,\ c_6,\ c_7$ satisfy $c_6(c_5 - 5c_7) > 0$ and their values are kept.

Proof of Theorem 1.1. Let $V_{2i-1}(O) = 0,\ i = 1, 2, 3, 4$, then we get $c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c_4 = -5c_6$ and $V_9(O) = -\frac{1}{128}\pi c_6 \varepsilon^2 (c_5 - 5c_7) < 0$. From the relationship between stability of focus O and the sign of its Lyapunov constants, we know that the origin O is a fine focus and stable. In the following, by using the similar disturbing technique to the one of [13], we prove that system (2) has 4 small amplitude limit cycles in the neighborhood of the origin.

First, we let c_4 satisfy $0 < -(c_4 + 5c_6)\varepsilon \ll \varepsilon^2$ and $c_1 = c_2 = c_3 = 0$, that is $V_{2i-1}(O) = 0,\ i = 1, 2, 3$ and $V_7(O) > 0$. So the stability of the origin has changed. By applying Poincarè–Bendixson Theorem, we know that there exists a stable small amplitude limit cycle around the origin.

Then we keep the value of c_4 and let c_3 satisfy $0 < -c_3\varepsilon \ll -(c_4 + 5c_6)\varepsilon \ll \varepsilon^2$ and $c_1 = c_2 = 0$, that is $V_1(O) = V_3(O) = 0$ and $V_5(O) < 0$. Then the stability of the origin has changed again. By applying the Poincarè–Bendixson Theorem again, we get the other unstable small amplitude limit cycle.

Next, we keep the value of $c_4,\ c_3$ and let c_2 satisfy $0 < c_2\varepsilon \ll -c_3\varepsilon \ll \varepsilon^2$ and $c_1 = 0$, that is $V_1(O) = 0,\ V_3(O) > 0$. Again, the origin changes its stability from stable to unstable. For the similar analysis, we know that there exists a third small amplitude limit cycle which is stable around the origin.

Finally, we keep the value of $c_4,\ c_3,\ c_2$ and let c_1 satisfy $0 < -c_1\varepsilon \ll c_2\varepsilon \ll \varepsilon^2$, that is the origin has changed its stability from unstable to stable. Due to the same reason, system (2) has the fourth small amplitude limit cycle in the neighborhood of the origin O.

From the continuous dependence of solution with respect to the parameters of the differential equation, we know that the small amplitude limit cycles are well kept as the parameters are slightly changed. Therefore, system (2) has 4 small amplitude limit cycles in the neighborhood of the origin O.

The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2. From Lemma 2.3 and by using similar disturbing technique to Theorem 1.1, we know that system (2) can have 4 small amplitude limit cycles in the neighborhood of singular point $A_1(\varepsilon)$. Noting that system (2) is Z_4 equivariant, we know that Theorem 1.2 holds. □

Proof of Theorem 1.3. Let parameters $c_1,\ c_2,\ c_3,\ c_4$ satisfy that $c_1 = \varphi_1(c_2, c_3, c_4, c_6, \varepsilon)$, $c_2 = \psi_1(c_3, c_4, c_6, \varepsilon)$, $c_3 = \psi_2(c_4, c_6, \varepsilon)$, $c_4 = \psi_3(c_6, \varepsilon)$, from Lemma 3.2, we know that system (2) has a saddle connection loop $\Gamma_{poly}(\varepsilon)$ and $div(S_1(\varepsilon)) = 0,\ \sigma(L_{S_1(\varepsilon)\widehat{\,} S_2(\varepsilon)}) = 0,\ R_1(S_1(\varepsilon)) = 0$.

In the following, we prove that under the above conditions the saddle connection loop $\Gamma_{poly}(\varepsilon)$ is isolated. Noting the analyticity of system (2), we only need to prove that the singular point O is a fine focus.

From Lemma 2.1, we get $V_{2i-1}(O) = o(\varepsilon),\ i = 1, 2, 3, 4, 5$. We claim that one of Lyapunov Constants $V_{2i-1}(O),\ i = 1, 2, 3, 4, 5$ does not equal to zero. If this claim is not true, then from $V_{2i-1} = 0,\ i = 1, 2, 3, 4$ and Lemma 2.1, we get $V_9(O) = -\frac{1}{128}\pi c_6\varepsilon^2(c_5 - 5c_7) < 0$ which contradicts with $V_9(O) = 0$. Therefore, singular point O is a fine focus and $\Gamma_{poly}(\varepsilon)$ is isolated.

As to the inner stability of $\Gamma_{poly}(\varepsilon)$, there are two possibilities:

(i) $\Gamma_{poly}(\varepsilon)$ is inner stable;
(ii) $\Gamma_{poly}(\varepsilon)$ is inner unstable.

Then we prove that in the above two cases, system (2) can have 4 limit cycles which are near inner side of $\Gamma_{poly}(\varepsilon)$. For similarity, we only give the proof of result in case (i).

Firstly, we choose c_4 to satisfy $0 < -(c_4 - \psi_3(c_6, \varepsilon)) \ll \varepsilon^2$ and let $c_1 = \varphi_1(c_2, c_3, c_4, c_6, \varepsilon)$, $c_2 = \psi_1(c_3, c_4, c_6, \varepsilon)$, $c_3 = \psi_2(c_4, c_6, \varepsilon)$. That is $div(S_1(\varepsilon)) = \sigma(L_{S_1(\varepsilon)\widehat{\,} S_2(\varepsilon)}) = 0$ and $R_1(S_1(\varepsilon)) > 0$. From Lemma 3.3, we know that $\Gamma_{poly}(\varepsilon)$ has changed its stability from stable to unstable. By using the Poincarè–Bendixson theorem, we get the first limit cycle near $\Gamma_{poly}(\varepsilon)$.

Secondly, we fix the value of c_4 and choose c_3 to satisfy $0 < c_3 - \psi_3(c_4, c_6, \varepsilon) \ll |c_4 - \psi_3(c_6, \varepsilon)|$ and continue to let $c_1 = \varphi_1(c_2, c_3, c_4, c_6, \varepsilon)$ and $c_2 = \psi_1(c_3, c_4, c_6, \varepsilon)$. Then $div(S_1(\varepsilon)) = 0,\ \sigma(L_{S_1(\varepsilon)\widehat{\,} S_2(\varepsilon)}) < 0$ and from Lemma 3.3, we know that $\Gamma_{poly}(\varepsilon)$ has changed its stability again. For the same reason, we get the second limit cycle near $\Gamma_{poly}(\varepsilon)$.

Thirdly, we fix the value of c_4, c_3 and choose c_2 to satisfy $0 < c_2 - \psi_1(c_3, c_4, c_6, \varepsilon) \ll |c_3 - \psi_2(c_4, c_6, \varepsilon)|$. That is $dis(S_1(\varepsilon)) > 0$. Then from Lemma 3.3, we know that $\Gamma_{poly}(\varepsilon)$ has changed its stability from stable to unstable. Using the similar analysis, we get the third limit cycle.

Finally, we fix the value of c_4, c_3, c_2 and choose c_1 to satisfy $0 < c_1 - \varphi_1(c_2, c_3, c_4, c_6, \varepsilon) \ll |c_2 - \psi_1(c_3, c_4, c_6, \varepsilon)|$. That is $d(\varepsilon, L_{\widehat{S_1 S_2}}) < 0$. Then from Lemma 3.1, we know that saddle connection $L_{S_1(\varepsilon)\widehat{\,} S_2(\varepsilon)}$ has been broken. By using the Poincarè–Bendixson theorem again, we get the fourth limit cycle near inner side of $\Gamma_{poly}(\varepsilon)$.

The proof of the Theorem 1.3 is completed. □

Proof of Theorem 1.4. Let $c_1 = \varphi_2(c_2, c_3, c_4, c_6, \varepsilon)$, $c_2 = \psi_4(c_3, c_4, c_6, \varepsilon)$, $c_3 = \psi_5(c_4, c_6, \varepsilon)$, $c_4 = \psi_6(c_6, \varepsilon)$, then from Lemma 3.4 we know system (2) has a homoclinic loop $\Gamma_{S_1(\varepsilon)}$ and its stability quantities satisfy $div(S_1(\varepsilon)) = 0$, $\sigma(\Gamma_{S_1(\varepsilon)}) = 0,\ R_1(S_1(\varepsilon)) = 0$.

By applying the similar analysis, we can prove that the homoclinic loop $\Gamma_{S_1(\varepsilon)}$ is isolated. Further, by disturbing the value of the parameters $c_4,\ c_3,\ c_2,\ c_1$ slightly to change the stability of $\Gamma_{S_1(\varepsilon)}$ till $\Gamma_{S_1(\varepsilon)}$ breaks, then 4 limit cycles appear near the inner side of $\Gamma_{S_1(\varepsilon)}$.

The proof of the theorem is completed. □

Proof of Theorem 1.5. Let $c_1 = \varphi_3(c_3, c_4, c_6, \varepsilon)$, $c_2 = \varphi_4(c_3, c_4, c_6, \varepsilon)$, $c_3 = \psi_7(c_4, c_6, \varepsilon)$ and $c_4 = \psi_8(c_6, \varepsilon)$, then from Lemma 3.5 we know that system (2) has a compound cycle $\Gamma_{comp}(\varepsilon)$ and its outer stability quantities satisfy $div(S_1(\varepsilon)) = 0$, $\sigma(\Gamma_{comp}(\varepsilon)) = \sigma(\Gamma_{S_1(\varepsilon)}) + \sigma(L_{S_1(\varepsilon)\widehat{\,} S_2(\varepsilon)}) = 0$ and $R_1(S_1(\varepsilon)) = o(\varepsilon)$. We are not certain about the sign of $R_1(S_1(\varepsilon))$, but from Lyapunov constants of the origin and the same analysis as in the proof of Theorem 1.3, we know that compound cycle $\Gamma_{comp}(\varepsilon)$ is isolated.

As to the outer stability of $\Gamma_{comp}(\varepsilon)$, there are two possible cases:

(i) $\Gamma_{comp}(\varepsilon)$ is outer stable;
(ii) $\Gamma_{comp}(\varepsilon)$ is outer unstable.

For the similarity of the proof, here we only give only the proof of Theorem 1.5 in the case (i).

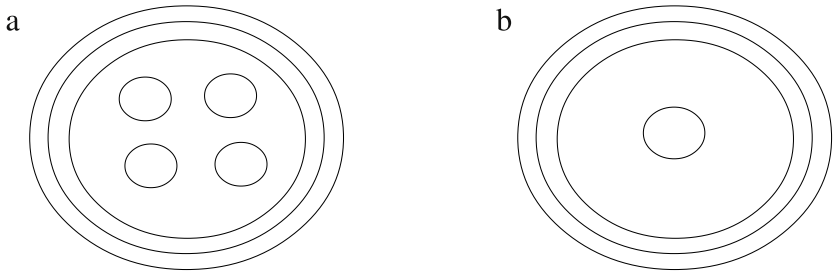

Fig. 2. The distributions of limit cycles of system (2) in Theorem 1.5.

Suppose $\Gamma_{comp}(\varepsilon)$ is outer stable. Firstly, we choose c_4 to satisfy $0 < c_4 - \psi_8(c_6, \varepsilon) \ll \varepsilon^2$ and let $c_1 = \varphi_3(c_3, c_4, c_6, \varepsilon)$, $c_2 = \varphi_4(c_3, c_4, c_6, \varepsilon)$, $c_3 = \psi_7(c_4, c_6, \varepsilon)$. That is $\sigma(\Gamma_{comp}(\varepsilon)) > 0$. From Lemma 3.5, we know that system (2) still has a compound cycle Γ_{comp} and it is unstable. By applying Poincarè–Bendixson Theorem, we get one stable large limit cycle near the outer side of $\Gamma_{comp}(\varepsilon)$. Secondly, we fix the value of c_4 and make c_3 satisfy that $0 < c_3 - \psi_7(c_4, c_6, \varepsilon) \ll |c_4 - \psi_8(c_6, \varepsilon)|$ and let $c_1 = \varphi_3(c_3, c_4, c_6, \varepsilon)$, $c_2 = \varphi_4(c_3, c_4, c_6, \varepsilon)$. Under such conditions, for $div(S_1(\varepsilon)) < 0$, so compound cycle $\Gamma_{comp}(\varepsilon)$, homoclinic loop $\Gamma_{S_1(\varepsilon)}$ and saddle connection loop $\Gamma_{poly}(\varepsilon)$ are all stable. That is $\Gamma_{comp}(\varepsilon)$ has changed its outer stability again. By using Poincarè–Bendixson Theorem again, we get the other unstable large limit cycle near the outer side of $\Gamma_{comp}(\varepsilon)$. Thirdly, we fix the value of c_4, c_3 and let c_2 change slightly to satisfy $d(\varepsilon, L_{\widehat{S_1 S_2}}) < 0$ and let c_1 satisfy $d(\varepsilon, \Gamma_{S_1}) = 0$, in other words, saddle connection $L_{\widehat{S_1(\varepsilon) S_2(\varepsilon)}}$ has broken while homoclinic loop $\Gamma_{S_1(\varepsilon)}$ is kept. From similar reason, system (2) has third large limit cycle near the outer side of $\Gamma_{comp}(\varepsilon)$ which has broken. Finally, we fix the value of c_4, c_3, c_2 and change c_1 to satisfy $d(\varepsilon, \Gamma_{S_1}) > 0$, that is homoclinic loop $\Gamma_{S_1(\varepsilon)}$ has broken. For the similar reason, we get a medium limit cycle near the inner side of $\Gamma_{S_1(\varepsilon)}$. Noting the Z_4-equivariance of system (2), we know the first part of Theorem 1.5 is true.

If in the above third step, we fix the value of c_4, c_3 and choose c_2 to satisfy $d(\varepsilon, \Gamma_{S_1}) < 0$ and let c_1 satisfy $d(\varepsilon, L_{\widehat{S_1 S_2}}) = 0$, we get the third large limit cycle. Then we choose c_1 slightly to satisfy $d(\varepsilon, L_{\widehat{S_1 S_2}}) > 0$, we get medium limit cycle near the inner side of $\Gamma_{poly}(\varepsilon)$ which has broken. Noting $\Gamma_{comp}(\varepsilon)$ circling around all the singular points of system (2), we know the conclusions of Theorem 1.5 is true.

The proof of the Theorem 1.5 is completed. □

Remark 2. The distribution of 7 limit cycles (res 4 limit cycles) of system (1) in Theorem 1.5 are shown in Fig. 2. As we know, the configuration of limit cycles in case (a) in Fig. 2 is new.

5. Conclusions

In this paper, the qualitative method of differential equation is used to study the number and distribution of limit cycles of a perturbed quintic Hamiltonian system (1). The existence and stability theory of heteroclinic loop, homoclinic loop and compound cycle are applied to study the heteroclinic loop, homoclinic loop and compound cycle bifurcations of such system under Z_4-equivariant quintic perturbation. By combining first several Lyapunov constants of singular points of system (2), we find that the perturbed system (2) can have at least 4 limit cycles bifurcated from saddle connection loop $\Gamma_{poly}(\varepsilon)$ and have at least 3 large limit cycles and 4 medium limit cycles(or 1 medium limit cycle) bifurcated from the compound cycle $\Gamma_{comp}(\varepsilon)$.

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